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TILINGS AND DISCRETE DIRICHLET PROBLEMS

BY

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ABSTRACT

Let f be a harmonic function on a finite planar Markov chain M whose boundary consists of two vertices on the same face. We construct a geometric realization of (M, f) as a tiling of a rectangle with trapezoids, each trapezoid having two horizontal edges. Conversely, each such tiling arises in this way. Similar results hold for harmonic functions with more general boundary conditions.

Certain prescriptions of transition probabilities on edges in M give rise to tilings with prescribed shapes. This allows us to give necessary conditions for the existence of a tiling of an arbitrary polygon with squares, equilateral triangles, and so on. Using this method, we classify all polygons with at most one non-convex vertex which can be tiled with squares. A similar classification holds for tiling with equilateral triangles. We determine the Euclidean tori which can be square-tiled.

1. Introduction

In 1903, Dehn indicated a relationship between planar resistor networks (reversible Markov chains) and tilings of rectangles by squares [8]. By placing a potential across two vertices on the outer face of the network, the resulting potential field and current flow could be realized geometrically as a tiling of a rectangle by squares. He used the networks to get information about the possible

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tilings. In particular, he proved that a rectangle which can be tiled with squares has rational aspect ratio [8], see also [5].

In this article we generalize his construction. We consider *not necessarily reversible* planar Markov chains (i.e. planar graphs with transition probabilities); we show that to each such Markov chain with "boundary" consisting of two vertices on the same face there is naturally associated a tiling of a rectangle with trapezoids. By a **trapezoid** we mean here a quadrilateral with two *horizontal* edges.

The main result of the paper is the correspondence between trapezoid tilings of general polygons and harmonic functions on planar Markov chains. By a **harmonic function** on M, a Markov chain with boundary, we mean a function on the vertices of M which is harmonic except at the boundary vertices. The following table describes this correspondence:

Trapezoid tiling	harmonic function
tile	edge
maximal horizontal edge	vertex
maximal non-horizontal edge	face
aspect ratios of tiles	stationary measure
y-coordinate of horiz. edge	harmonic function on vertices
slope of an edge	winding number of random walk
boundary vertices	Markov chain boundaries
area	Dirichlet energy

For the explanations of these correspondences see below.

An application of this correspondence is a generalization of Dehn's result: in Theorem 6, we prove a rationality condition for polygons which are tiled by "rational" shapes.

In the special case of tilings with squares or with equilateral triangles, we have a stronger criterion for tilability: we associate to a polygon P a quadratic form which must be symmetric and positive semidefinite in order for there to exist a tiling of P (Theorems 12 and 16).

As a concrete application of these results, we classify polygons with at most one nonconvex vertex which can be tiled with squares or equilateral triangles. The case of tilings of *convex* polygons was solved for squares by Dehn [8] and for equilateral triangles by Tutte [4]. A related argument allows us to classify the square-tilable Euclidean tori.

Another application which we do not discuss here is to the existence of bounded harmonic functions on infinite planar Markov chains [1], which is a generalization of the result of Benjamini and Schramm [2]. In this application, the tiling gives information about the Markov chain.

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2. Background and notation

For background in random walks and Markov chains, see [9, 10].

A **graph** G is a pair $G = (V, E)$, where V is a finite set of vertices and $E \subset V \times V$ is a symmetric set ($(i,j) \in E$ iff $(j,i) \in E$). If $(i,j) \in E$ we say i and j are **neighbors**.

A **Markov chain** is a pair (G, e) consisting of a *connected* graph G and a function $e\colon V \times V \to [0,1]$ (the transition probabilities) which is 0 on the complement of E and satisfies, for each i:

$$\sum_{j\in V} e(i,j) = 1. \tag{1}$$

A **Markov chain with boundary** is a triple (G, e, B) where (G, e) is a Markov chain and $B \subset V$. The set B is the set of **boundary vertices**, or **nodes**.

By a **random walk on** (G, e) **starting at vertex** i we mean a sequence of random variables $\{x_0, x_1, x_2, \ldots\}$, $x_i \in V$, satisfying $x_0 = i$ and

$$\mathbb{P}(x_{n+1} = j \mid x_n = i) = e(i,j).$$

Thus the $e(i,j)$ are transition probabilities for the random walk. We say (G, e) is **ergodic** if for any two vertices i, j, a random walk started at i eventually reaches j with probability 1.

Let $m\colon V \to [0, \infty)$ be a stationary distribution for the random walk, that is, for all i we have

$$m(i) = \sum_{j\in V} m(j)e(j,i). \tag{2}$$

If we assume G is ergodic then m is unique up to scale: $m(v)$ is proportional to the average amount of time that an infinite random walk will spend at v. Generally the dimension of the space of distributions m is the same as the number of ergodic components (sinks) of (G, e).

Define the **stationary distribution on edges** m: $V \times V \to [0,\infty)$ by $m(i,j) = m(i)e(i,j)$. Then by (1) m satisfies

$$\sum_j m(i,j) = \sum_j m(j,i). \tag{3}$$

Note that if G is ergodic then $m(i,j)$ is the average fraction of crossings of the edge (i,j) from i to j, for an infinite random walk.

Let $a \neq b$ be any two vertices of (G,e). Define a function f: $V \to \mathbb{R}$ by: $f(i)$ equals the probability that a random walk started at i will reach a before reaching b. Then f satisfies $f(a) = 1$, $f(b) = 0$ and for $i \neq a, b$

$$f(i) = \sum_j f(j)e(i,j). \tag{4}$$

We say that any function f: $V \to \mathbb{R}$ satisfying (4) is **harmonic at** i. A function is **harmonic** for (G,e,B) if it is harmonic at every vertex of $V - B$. By (4), harmonic functions obey a maximum principle: the maximum and minimum value of any harmonic function occurs in B. This implies that a harmonic function is determined by its values on B. In particular the vector space of harmonic functions on $(G,e,\{a,b\})$ is spanned by the constant function and f, the function constructed above.

3. Planar Markov chains

We now assume that G is a planar graph, embedded in the plane. A **face** of G is a connected component of the complement of G in the plane. The **outer face** is the unbounded face. Let F denote the set of faces of G, with $o \in F$ denoting the outer face. Two faces $x, x' \in F$ are said to be **adjacent** if they share a common edge (i,j). Because of the planar embedding, given an oriented edge (i,j) there is an order to the two faces x, x' sharing that edge: on a path from i to j one of x or x' will be on the left. In case x is the one on the left, we write $(x,x') = *(i,j)$. Thus $*$ is a map from E to ordered pairs of adjacent faces.

Let (G,e) be a planar Markov chain, with G embedded in the plane.

We define a function w: $F \to \mathbb{R}$ satisfying: $w(o) = 0$, and for any two adjacent faces x, x' with $(x,x') = *(i,j)$ we have

$$w(x) - w(x') = m(i,j) - m(j,i). \tag{5}$$

These two properties define a unique function: the value at any face x can be determined by taking a path of adjacent faces from o to x and integrating the equations (5). By the property (3) this value is independent of the path taken: any closed path of faces integrates to 0.

Yuval Peres gave the following probabilistic interpretation of the function w, assuming ergodicity of (G, e). The value $w(x)$ is the expected relative winding number around the face x of a long random walk. That is, if we take a long but finite random walk on G, starting at any vertex v, and close up the walk with an arbitrary choice of path of bounded length, then the (counterclockwise) winding number of the walk around the face x divided by the total path length will be proportional to $w(x)$. To see this it suffices to note that for two adjacent faces the difference in their winding numbers is exactly the expected signed number of crossings of the edge (i, j) between them by a random walk. This is exactly the difference $m(i, j) - m(j, i)$.

4. The Markov chain associated to a trapezoid tiling

By a **trapezoid** we will mean a polygon whose vertices are an (ordered) 4-tuple $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$ of points in $\mathbb{R}^2$ satisfying

$$x_2 \geq x_1,$$
$$x_3 \geq x_4,$$
$$y_4 = y_3 > y_1 = y_2,$$

where at most one of the $\geq$ can be an equality. Thus a trapezoid is a trapezoid in the usual sense oriented so that the parallel edges are parallel to the x-axis, except that we allow one or the other of the parallel edges to be reduced to a point, so that a trapezoid can be a triangle. The two "edges" $\{(x_1, y_1), (x_2, y_2)\}$ and $\{(x_3, y_3), (x_4, y_4)\}$ are called the **lower** and **upper** edges of the trapezoid, respectively. For a polygon $P \in \mathbb{R}^2$ a **trapezoid tiling** is a tiling of P with trapezoids. More precisely, it is a finite set of trapezoids whose union is P and whose interiors do not intersect. Note that any polygon has a trapezoid tiling.

We show how to associate a harmonic function on a planar Markov chain to a trapezoid tiling T of a polygon P. Let $G = (V \cup \{\infty\}, E)$ be the graph where V is indexed by the set of connected components of the union in $\mathbb{R}^2$ of all the upper and lower edges of trapezoids in the tiling, and ∞ is an additional vertex. Let B be the union of $\{\infty\}$ and the set of vertices in V whose corresponding components contain at least one vertex of P.

There is an edge in G for each tile in T: for a tile t let i and j be the vertices in V corresponding to the components containing t's upper and lower edges respectively. Put edges $(i, j), (j, i)$ in G with weights $m(i, j) = t_u/h,\ m(j, i) =$

t_l/h, where t_u is the length of the upper edge of t, t_l the length of the lower edge of t, and h is the height of t (difference in y-coordinates of its upper and lower edges). Also put edges (b, ∞) for all $b \in B - \{\infty\}$, with weights determined as follows. Consider the two non-horizontal edges e_1, e_2 (in counterclockwise order) of P adjacent to the edge (or vertex) corresponding to b. Let s_1, s_2 be the slopes of e_1, e_2, respectively. Take $m(b, \infty), m(\infty, b)$ non-negative and so that $m(b, \infty) - m(\infty, b) = 1/s_1 - 1/s_2$. See Figure 1.

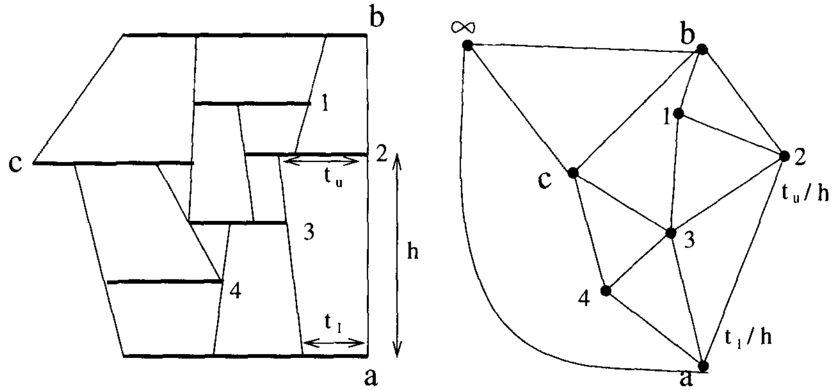

Figure 1.

From the function $m: E \to \mathbb{R}$ one can obtain the transition probabilities e by:

$$e(i,j) = \frac{m(i,j)}{\sum_j m(i,j)}.$$

Define a function $f: V - \{\infty\} \to \mathbb{R}$ by $f(i) =$ the y-coordinate of the component corresponding to vertex i.

A face of the graph G corresponds in the tiling to a line segment contained in the union of the non-horizontal boundaries of trapezoids, which is maximal among such segments not passing through the interior of any horizontal segment. The faces of G passing through ∞ are non-horizontal sides of the polygon P.

Let F be the set of faces of G. Define a function $w: F \to \mathbb{R}$ by: $w(x) = 1/s$, where s is the slope of the corresponding line segment in T. For a vertical segment set $w(x) = 0$.

LEMMA 1: *The function $m(i,j)$ is a stationary distribution on edges for the Markov chain (G, e, B). The function w is up to an additive constant the expected winding number function.*

Proof: For a trapezoid $\{(x_1, y_1), \ldots, (x_4, y_4)\}$ with left and right edges corresponding to faces x' and x of G and upper and lower edges corresponding to vertices i and j respectively, we have

$$
\begin{aligned}
w(x') &= (x_4 - x_1)/(y_4 - y_1) = (x_4 - x_1)/(y_3 - y_1) \\
w(x) &= (x_3 - x_2)/(y_3 - y_2) = (x_3 - x_2)/(y_3 - y_1) \\
m(i,j) &= (x_3 - x_4)/(y_3 - y_1) \\
m(j,i) &= (x_2 - x_1)/(y_3 - y_1)
\end{aligned} \tag{6}
$$

and so $w(x) - w(x') = m(i,j) - m(j,i)$, which is (5). For an edge (b, ∞), we have (5) by construction. Summing around a vertex gives (3), implying that m is a stationary distribution. Since w is uniquely determined by m (given its value 0 on some face), this completes the proof. ∎

LEMMA 2: *The function f is a harmonic function for (G, e, B).*

Proof: Let $i \in V - B$ and $j_1, \ldots, j_k$ be the vertices adjacent to i. Assume that $f(j_r) \geq f(i)$ for $r \in [1, \ell]$ and $f(j_r) < f(i)$ for $r \in [\ell + 1, k]$. The trapezoids for edges (i, j_r) with $r \in [1, \ell]$ have lower edges at y-coordinate $f(i)$. These lower edges have lengths $m(i, j_r)(f(j_r) - f(i))$. The sum of these lengths is the length of the component i. The trapezoids for (i, j_r) with $r \in [\ell+1, k]$ have upper edges at y-coordinate $f(i)$: their upper edges have lengths $m(i, j_r)(f(i) - f(j_r))$. The sum of these is also equal to the length of the component i. So we have

$$\sum_{r \in [1,\ell]} m(i, j_r)(f(j_r) - f(i)) = \sum_{r \in [\ell+1,k]} m(i, j_r)(f(i) - f(j_r))$$

which is the harmonicity of f at i. ∎

It is clear that the graph G is planar: ignoring for the moment the vertex ∞, G can be drawn on top of the tiling T by placing each vertex at the midpoint of its corresponding component, and drawing the edges from a vertex as paths which run along the component until they lie over the corresponding tile and then up or down to the next component. (All the paths running along a component can be drawn disjoint except at the vertex). One can then put the vertex ∞ anywhere in the outer face and connect it to all other vertices in B, which are by construction on the outer face.

We have shown that to a trapezoid tiling of a polygon corresponds a harmonic function on a planar Markov chain (G, e, B). The Markov chains which arise have the property:

(7) *There is a vertex $\infty \in B$, whose neighbors are all the other vertices of B.*

Note that in case all the non-horizontal edges of P are parallel to each other, we can choose $m(b, \infty) = m(\infty, b) = 0$ for all b; so we can effectively ignore the vertex ∞ altogether. In this case B is contained in the vertices of a single face of G.

5. The tiling associated to a Markov chain

In general a harmonic function on a planar Markov chain with boundary will not give a nice trapezoid tiling; one must require some extra hypotheses even to have an immersed tiling (see below). However in the simplest case, when B consists of two vertices on a single face of G, we do get a tiling, a trapezoid tiling of a rectangle. This is what we now describe. (Comments on the more general cases are below.)

Let $(G, e, \{a, b\})$ be a planar Markov chain with boundary, with G embedded in $\mathbb{R}^2$. Assume a and b lie on the outer face of G.

Let $m: E \to \mathbb{R}$ be a stationary distribution on the edges, w the corresponding winding number function. Let f be the unique harmonic function satisfying $f(a) = 1$ and $f(b) = 0$.

There is a slight difficulty in the construction if some pair of adjacent vertices of G has the same f-value: such an edge would give rise to a trapezoid of zero area. Rather than generalize the definitions, we choose to **contract** these edges: if (i, j) is an edge of G and $f(i) = f(j)$, then we will replace $(G, e, \{a, b\})$ with the Markov chain $(G', e', \{a', b'\})$ in which we identify vertices i and j (creating a new vertex i'), and remove edge (i, j) (and $a' = a$, $b' = b$ unless a or b is one of the vertices in the contraction: in this case the new vertex i' will replace the lost boundary vertex). The new edges are those of G along with edges (i', k) and (k, i') for each neighbor k of i or j. Define $m'(i', k) = m(i, k) + m(j, k)$ and $m'(k, i') = m(k, i) + m(k, j)$ for all k. Define $f(i')$ to be the common value $f(i) = f(j)$. The values of w are unchanged in this process, although some faces of G may disappear.

An alternative method of dealing with this difficulty is to perturb the transition probabilities by some small amounts to break any such symmetries, and then take the limit of the tilings as the perturbations tend to 0. This would have essentially the same effect as the above contraction process.

Thus we will assume that $f(i) \neq f(j)$ for any edge (i, j).

For an edge (i, j) adjacent to faces x, x' the values

$$f(i), f(j), m(i, j), m(j, i), w(x), w(x')$$

determine a trapezoid up to horizontal translation in the following sense. Assume that $f(i) > f(j)$. Let T_{ij} be the trapezoid with vertices

$$\{ \begin{array}{rl} (0 & ,f(j)), \\ (m(j,i)(f(i)-f(j)) & ,f(j)), \\ ((w(x)+m(j,i))(f(i)-f(j)) & ,f(i)), \\ (w(x')(f(i)-f(j)) & ,f(i))\}. \end{array} \tag{8}$$

(Compare equation (6).) In a tiling corresponding to G, the tile corresponding to the edge (i,j) will be a horizontal translation of T_{ij}.

To determine the horizontal placements of the trapezoids, we use the following observation:

LEMMA 3: *The function f has no saddle points, that is, as j runs cyclically around the neighbors of a vertex $i \neq a, b$, the values of $f(i) - f(j)$ change sign exactly twice.*

Proof: This is a consequence of the planarity of G. Because f is harmonic, from any vertex i there is a connected path $i = i_0, i_1, \ldots, i_k = a$ in the graph such that $f(i_{\ell+1}) > f(i_\ell)$ for each ℓ. Similarly there is a path of decreasing f-values from i to b. If i is a saddle point, there are two increasing paths from i to a, and two decreasing paths to b, which start along edges interleaved cyclically around i. The increasing paths have union enclosing a region U which contains a vertex whose f-value is less than $f(i)$, but does not enclose b. This violates the maximum principle for f in the region U. ∎

We build the tiling from the bottom up, starting at vertex b and looking at increasing values of f.

The set of edges incident to b has a linear order, which is the cyclic order starting from the outer face and moving clockwise around b. Let $T_1(b), \ldots, T_k(b)$ be the corresponding set of trapezoids. Place the first trapezoid $T_1(b)$ so that its lower left corner is at the origin. For each ℓ place the trapezoid $T_{\ell+1}(b)$ so that its lower left vertex touches the lower right vertex of $T_\ell(b)$. Note that the slope of the left edge of $T_{\ell+1}(b)$ equals the slope of the right edge of $T_\ell(b)$ (since the edges bound the same face of G), so that the union of these trapezoids covers a neighborhood of the appropriate part of the x-axis, and there is no overlap near the x-axis. Let x_0 be the x-coordinate of the right edge of $T_k(b)$; this right edge is vertical since it is incident on the outer face of G for which $w(o) = 0$.

We now proceed to build the tiling by induction. At a given time t we have a set of tiles which covers the region delimited by the x-axis and the line $y = t$, and by the y-axis and the line $x = x_0$. The interiors of the tiles do not overlap in this region. This partial tiling has the property that tiles coming from edges in G which share a common face in G have right or left boundaries which lie on a common non-horizontal line segment in the partial tiling.

We increase t until the line $y = t$ passes through the upper edges of one or more trapezoids in our partial tiling: each such trapezoid has upper edge which corresponds to a vertex i of G with $f(i) = t$. By induction we have already included in our tiling trapezoids for each edge (i, j) for which $f(j) < f(i)$. Furthermore tiles for these edges are adjacent, one to the next, by the comment at the end of the previous paragraph. Let $T_1(i), \ldots, T_{k_i}(i)$ be these tiles, so that $T_{\ell+1}(i)$ is adjacent and to the right of $T_\ell(i)$. Let $T^1(i), \ldots, T^{k'_i}(i)$ be tiles corresponding to edges (i, j) with $f(j) > f(i)$, in cyclic order clockwise around i (by Lemma 3 these occur consecutively). Place all the $T^\ell(i)$ so that the lower left corner of $T^1(i)$ is at the upper left corner of $T_1(i)$, and each $T^{\ell+1}(i)$ has its lower left corner at the lower right corner of $T^\ell(i)$.

By the harmonicity of f (see Lemma 2), the sum of the lengths of the lower edges of the $T^\ell(i)$ equals the sum of the lengths of the upper edges of the $T_\ell(i)$. This implies that the upper tiles $\{T^\ell(i)\}$ fit exactly into the space over the lower tiles at i. The slope of the left edge of $T^1(i)$ equals the slope of the left edge of $T_1(i)$, and similarly for the right edges of $T^{k'_i}(i)$ and $T_{k_i}(i)$, since their edges in G share common faces.

Thus our partial tiling extends past t and does not overlap near t.

We continue increasing t, inserting tiles when t passes through the value of $f(v)$ for any vertex v. When t gets to 1, all tiles have been inserted; the remaining tiles with upper edge at 1 all come from edges adjacent to a, and the adjacency relation is the same. Thus we finally have a tiling of an $x_0 \times 1$ rectangle.

In order for the above construction to work for harmonic functions with more complicated boundary conditions, we have two necessary conditions: first, either (7) holds or B must be a subset of a single face of G. Secondly, Lemma 3 must be satisfied: the harmonic function f must have no saddle points. Even these two conditions are not sufficient; it may happen that the tiling will be an "immersed" tiling rather than an embedded tiling (that is, it may be a tiling of a locally Euclidean disk, with a locally isometric non-injective immersion to the plane). One can make such a tiling by taking copies of all the trapezoids for the edges in G, as defined by (8), except for edges (b, ∞), and abstractly gluing together the appropriate edges at each vertex and face. The gluing is well-defined by the cyclic order at each vertex of G (given by the embedding of G). The metric space that results from this gluing is a topological disk with a Euclidean structure, which may or may not be embeddable isometrically.

In the special case when f has exactly one local maximum and one local minimum in cyclic order around the face containing B, the disk is isometrically embeddable in $\mathbb{R}^2$. In this case the polygon P will be horizontally convex, that is, the intersection of P with any horizontal line is convex. Conversely, any horizontally convex P with a trapezoid tiling will give rise to a harmonic function with one local maximum and one local minimum.

6. Special choices of weights

The origins of the constructions above arose not from considering Markov chains but from considering tilings with restricted shapes, in particular squares or equilateral triangles. In 1903 Dehn [8] showed the relation between square tilings and electrical networks with resistances 1 on each edge. Brooks et al indicate in [4] a corresponding result for tilings with equilateral triangles.

Here we indicate how one might arrange the transition probabilities so that the resulting tiling has the desired shapes.

It is not obvious in what way the choice of transition probabilities (or rather, the weights m) on the edges of G determine the shapes of the tiles: one must first compute the expected winding number w. However in special cases we can determine w in advance:

6.1 Rectangles. It is clear from the definition that, for a trapezoid tiling in which some tile t is a rectangle with aspect ratio $height/width = R$, then in the corresponding Markov chain its edge will have weights $m(i,j) = m(j,i) = 1/R$. Conversely, if $m(i,j) = m(j,i)$ then the tile for edge (i,j) will be a parallelogram with aspect ratio $1/m(i,j)$.

Proposition 4 ([8]): *If the Markov chain $(G, e, \{a,b\},)$ (with a, b on the outer face) is reversible, that is, if for all i, j, $m(i,j) = m(j,i)$, then the corresponding tiling is by rectangles.*

A reversible Markov chain is called an **electrical network**; the weights $m(i,j)$ are the **edge conductances**. The winding number function w is identically zero. The harmonic function f is called the **potential**, and for an edge (i,j), $f(i)-f(j)$ is the **current** along (i,j) which satisfies the well-known Kirchhoff equations, of which (3) and (4) are a generalization.

Note that the choice of conductance 1 for each edge gives a tiling with squares.

6.2 Equilateral triangles. Let us construct the Markov chain from a tiling of a rectangle with isosceles right triangles, each of which has a horizontal side, a vertical side and a side of slope 1. Then each edge in G will have exactly one of its two weights $m(i,j)=0$, and the other weight $m(j,i)=1$. Furthermore the 1 and 0 weights alternate around any cycle (equivalently, at a vertex of G the weights on outgoing edges alternate). In particular the graph has bipartite dual. The faces have weights w alternatively 1 and 0. Conversely, we have

PROPOSITION 5: *If every vertex of G has even degree, and weights are chosen so that ($m(i,j)=1$ iff $m(j,i)=0$), the weights alternating around any vertex, then the tiling of $(G,e,\{a,b\})$ is a tiling of a rectangle with isosceles right triangles having edges of slopes $0,1,\infty$.*

By applying a global linear map $(x,y)\to(x-\frac{y}{2},\frac{\sqrt{3}y}{2})$, each tile becomes an equilateral triangle, and we get a tiling of a parallelogram with equilateral triangles.

6.3 EXAMPLE. Here is a third example. In Figure 2, we chose a graph G with bipartite dual. Each shaded face is assigned $w=\pm 1$ arbitrarily and each unshaded face is assigned $w=0$. By (5), these values of w determine m on condition that on each edge (i,j) one of $m(i,j)$ or $m(j,i)$ is zero. In this way we get a tiling of a rectangle by isosceles right triangles. Conversely, every isosceles right triangle tiling has such a Markov chain.

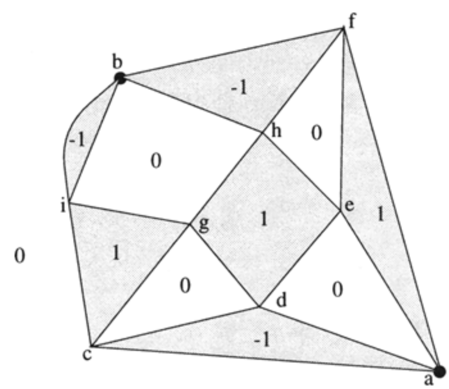

Figure 2. The Markov chain G; values of w.

7. An application: rationality

Let P be a polygon with a vertex at the origin. Suppose P is tiled with trapezoids, where each trapezoid has rational aspect ratio in the sense that the ratios t_u/h and t_l/h are rational (recall that t_u, t_l are the lengths of the upper and lower edges of the trapezoid and h is its height). Assume there is no horizontal line (in the union of the tile boundaries) connecting two non-adjacent vertices of P.

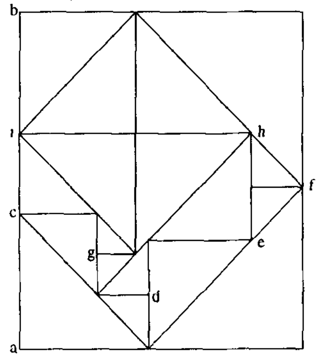

Figure 3. The tiling associated to the Markov chain in Figure 2.
Note that vertices h and i have the same f-value.

Then for the Markov chain associated to the tiling of P, the edge weights $m(i,j)$ for $i,j \neq \infty$ will all be rational numbers. Assume, by applying a linear shear $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ to P if necessary, that some edge e of P has rational slope. By the assumption at the end of the previous paragraph, each face of G is connected by a path of faces, not crossing any edge (b,∞), to a face containing e. Thus the winding number function will be rational, since it is uniquely determined on all edges by the $m(i,j)$ for $i,j \neq \infty$ (see (5)). So all the slopes of the sides of the trapezoids are rational (and all *edges of P* will have rational slope).

The harmonic function f is uniquely determined by the linear equations (4) and its boundary values, which are the y-coordinates of the vertices of P. Since the linear equations (4) have rational coefficients, the values of f are in the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by the y-coordinates of the vertices of P. This implies that the differences in x-coordinates of the vertices of P are in this same $\mathbb{Q}$-subspace. We therefore have

THEOREM 6: *Let P be a polygon with a vertex at the origin and an edge of rational slope. Assume that no two non-adjacent vertices of P have the same y-coordinate. If P can be tiled with trapezoids of rational aspect ratio, then all edges in the tiling have rational slope, and the x and y-coordinates of all vertices in the tiling are in the $\mathbb{Q}$-subspace of $\mathbb{R}$ spanned by the y-coordinates of vertices of P.*

The assumption on non-adjacent vertices is to preclude the existence of a horizontal edge in the tiling connecting non-adjacent vertices. One can do without this hypothesis but the conclusion is then more complicated to state.

A special case of the theorem is:

COROLLARY 7: *If a trapezoid P has a tiling with trapezoids of rational aspect ratio, then P has rational aspect ratio.*

This corollary extends a result of Dehn [8] who dealt with square tilings of a rectangle, and Tutte [4] who dealt with tilings with equilateral triangles. A proof of a closely related result by very different methods is given in [11].

8. Non-convex polygons

After the work of Dehn and Tutte, it is natural to ask which polygons P can be tiled with squares, and which can be tiled with equilateral triangles. In general this problem seems difficult: we give some indication about how one might solve the general problem below. Apparently the complexity increases with the number of vertices of P. Here we solve the simplest case where P is non-convex (the convex cases for squares and triangles were dealt with by Dehn and Tutte, respectively).

8.1 SQUARES. In this section we discuss tilings of rectilinear polygons (polygons having edges parallel to the axes) with squares. Let P be a rectilinear polygon with k horizontal edges, tiled with squares. According to the above, to such a tiling is associated a reversible planar Markov chain (i.e. resistor network) with boundary B consisting of k vertices (on the outer face). The edge conductances $m(x,y) = m(y,x)$ are all 1.

For a resistor network M with k boundary nodes, the **response** is the function L_M which gives the outgoing currents at the k nodes as a function of the potentials imposed at those nodes. For the square tiling, the currents are the *lengths* of horizontal edges, so the response gives the lengths of the horizontal edges (as rational linear functions of the y-coordinates of the vertices, as in Theorem 6). The response L_M is a symmetric linear mapping [6]. The work of Colin de Verdière, Gitler, Vertigan [6, 7] classifies planar resistor networks up to the equivalence relation of having the same response function:

THEOREM 8 ([6, 7]): *For any planar network G with k boundary nodes, there is a planar network G' (with real, positive edge conductances) with at most $k(k-1)/2$ edges having the same response. The network G can be reduced to G' by using a sequence of local rearrangements (called elementary transformations, as shown in Figure 4). Two planar networks having the same response and the* *minimal number of edges (among those of the same response) can be transformed into one another using only the $Y - \Delta$ move and its inverse (Figure 4).*

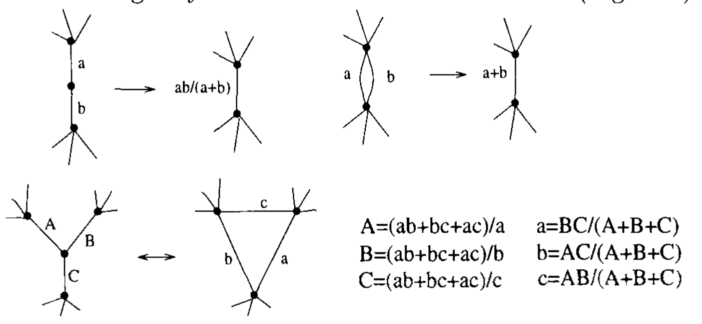

Figure 4. The labels on the edges are the conductances. Two elementary transformations not shown consist in: removing a self-loop at a vertex, and removing a non-boundary vertex of degree 1.

Note in Figure 4 that if the conductances are rational numbers then the new conductances, after applying a transformation, are also rational.

Suppose P is a rectlinear hexagon with side lengths $a + b, c, b, d, a, c + d$ in cyclic order starting from a horizontal edge (see Figure 5).

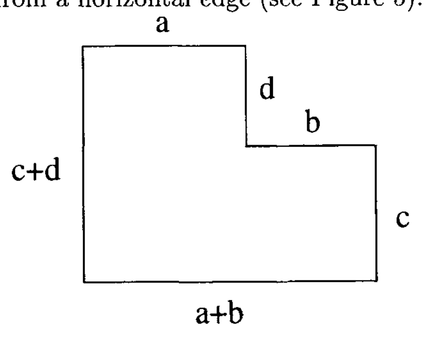

Figure 5.

Theorem 8 shows that any network with three nodes is equivalent to a triangle network, that is, the complete graph on the three nodes. The square-tiling of P (if it exists) can therefore be converted, using the local transformations in Figure 6, into a tiling using at most three rectangles. Furthermore, since the conductances were originally rational, the aspect ratios of these new rectangles will be *rational*. So rather than search for a general square tiling we can search for one of a very specific form: consisting of three rational rectangles placed as in Figure 7.

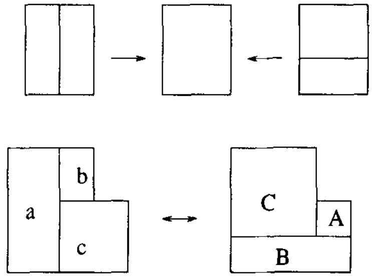

Figure 6. Square-tiling transformations corresponding to the three network transformations.

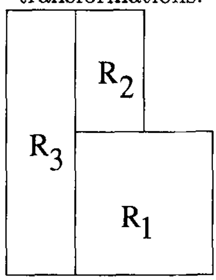

Figure 7. If P can be tiled then it can be tiled with three rectangles of rational aspect ratio as shown.

If the three rectangles R_1, R_2, R_3 in that figure have aspect ratios α, β, γ, then we must solve:

$$\begin{aligned} \gamma t &= c+d, \\ \alpha(a+b-t) &= c, \\ \beta(a-t) &= d, \end{aligned} \tag{9}$$

where t (the width of R_3) is real, and α, β, γ must be rational.

But Theorem 6 shows that a and b must be rational linear combinations of c and d. If c and d are not rationally independent (i.e. if $c \in \mathbb{Q}d$) then all sides are integer multiples of a single length, and so P can be tiled. Otherwise we can write

$$\begin{aligned} a &= \eta_1 c + \eta_2 d \\ b &= \eta_3 c + \eta_4 d \end{aligned} \quad \text{where } \eta_i \in \mathbb{Q}. \tag{10}$$

Then (9) has the unique solution

$$\begin{aligned} \alpha &= \frac{1+\eta_1}{\eta_1+\eta_3}, \\ \beta &= \frac{1}{\eta_2-\eta_1}, \\ \gamma &= \frac{1}{\eta_1}, \\ \eta_4 &= \eta_1-\eta_2. \end{aligned} \tag{11}$$

Here α, β, γ can take the value ∞ if one of the denominators $\eta_1, \eta_1 + \eta_3, \eta_2 - \eta_1$ is zero. On the other hand it is not possible that $\eta_1 = -1$ and $\eta_3 = 1$ simultaneously if we assume a, b, d are positive, so α, β, γ are all well-defined in $\mathbb{Q} \cup \{\infty\}$.

In conclusion,

THEOREM 9: *There is a square tiling of P if and only if either: all of a, b, c, d are integer multiples of a single length, or:* (10), (11) *hold with $\alpha, \beta, \gamma \in \mathbb{Q}_+ \cup \{\infty\}$ (nonnegative rationals or infinite).*

The equation $\eta_4 + \eta_2 - \eta_1 = 0$ is a consequence of the fact that the response must be symmetric. The remaining conditions are inequalities.

One can attempt the same method for polygons with more sides. If a rectilinear octagon can be tiled then it can be tiled with at most 6 rational rectangles, in, for example, the arrangement in Figure 8 (see [7]).

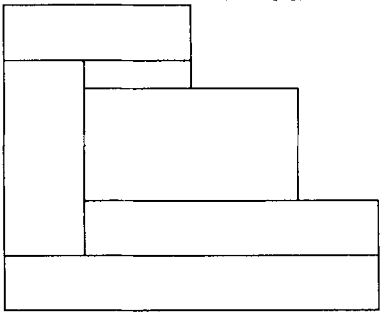

Figure 8. An octagon can be tiled with at most 6 rational rectangles if at all.

In this and all other cases we have equations for the response to be symmetric, and then some inequalities for the aspect ratios to be nonnegative. Colin de Verdière [6] nicely describes the set of possible responses for planar graphs by listing all the necessary inequalities. However, they are non-linear inequations and it is not clear whether the existence of a rational solution is decidable.

8.2 EQUILATERAL TRIANGLES. The non-convex polygon with the fewest sides and with angles which are multiples of $\pi/3$ is a pentagon. Let P be a pentagon with angles $\pi/3, \pi/3, 2\pi/3, 4\pi/3, \pi/3$ as in Figure 9 and side lengths $a + b + c, a, b, c, a + c$ as indicated. There are two other types of non-convex pentagons with angles multiples of $\pi/3$, but they can be obtained from P by taking b or c "less than zero".

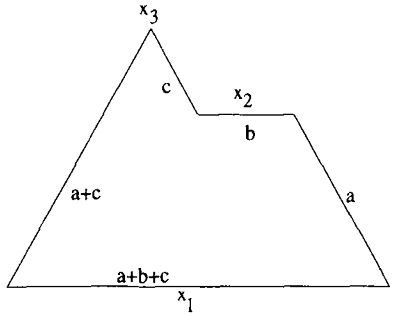

Figure 9.

A tiling of P with equilateral triangles gives a Markov chain (G, e, B) as in section 6.2, with three boundary nodes $B = \{x_1, x_2, x_3\}$ as indicated in the figure, where each edge has weights $m(i,j) = 1, m(j,i) = 0$ (and these weights alternate 0-1 around each interior vertex). There is a single edge incident to the node x_3 whose weight there must be zero.

By Theorem 6, if P is tilable we can write b as a rational linear combination of a and c:

$$b = \eta_1 a + \eta_2 c, \quad \eta_1, \eta_2 \in \mathbb{Q}.$$

We can assume that $a/c \notin \mathbb{Q}$ since otherwise it is trivial to tile. So η_1 and η_2 are uniquely determined.

We will prove the following inequalities, which follow from the maximum principle for a harmonic function on the Markov chain M:

$$\begin{aligned} \eta_2 &\leq 0, \\ \eta_2 &\geq -1, \\ \eta_1 &\geq 0. \end{aligned} \tag{12}$$

To prove the first inequality, look at the harmonic function which is zero on nodes x_1 and x_2, and equal to 1 on node x_3 (this corresponds to setting $a = 0, c = 1$).

The value of the function on all vertices in G, in particular those adjacent to x_2, must have nonnegative value. This implies that $b \leq 0$ when $a = 0, c = 1$, giving the first inequality. The value of this function on vertices adjacent to x_1 must also be nonnegative, and so $a+b+c \geq 0$. Since $a = 0$, this gives the second inequality.

For the third inequality, take the harmonic function which is 0 at x_1, and 1 and x_2 and x_3 (this corresponds to $c = 0, a = 1$). Then by the maximum principle the vertices adjacent to x_2 have value at most 1 and so $b \geq 0$.

PROPOSITION 10: *The pentagon P is tilable with equilateral triangles if and only if either: both $a/b, b/c \in \mathbb{Q}$, or $b = \eta_1 a + \eta_2 c$ with $\eta_1, \eta_2 \in \mathbb{Q}$ satisfying* (12).

Proof: It remains to construct a tiling. This is trivial if $a/b \in \mathbb{Q}$ and $b/c \in \mathbb{Q}$ since then the edges are all multiples of a single length.

In the second case, we can assume η_1, η_2 are not both 0. Let

$$\alpha = \frac{-\eta_2}{\eta_1 - \eta_2} \in [0,1] \cap \mathbb{Q},$$

and

$$\beta = \frac{\eta_2 + 1}{\eta_1 - \eta_2} \in [0, \infty) \cap \mathbb{Q}.$$

Then there is a tiling of P as in Figure 10 with convex shapes each having rational edge ratios. Each of these shapes has a tiling with equilateral triangles. ∎

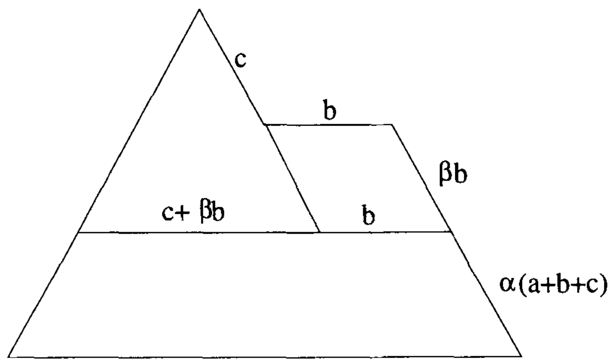

Figure 10. The tiling of a pentagon.

Note that the conditions in Proposition 10 are also the conditions for P to be tilable with rational trapezoids.

As in the case of squares, one can attempt to describe all the inequalities which the "response" of a planar Markov chain satisfies. In particular there are the inequalities which are generalizations of those of [6]. We don't know, however, if these are sufficient.

9. The Dirichlet energy

One aspect of a harmonic function we have not dealt with is the Dirichlet energy, which is defined to be

$$\sum_{edges} \frac{m(x,y) + m(y,x)}{2} (f(x) - f(y))^2,$$

the sum being over all edges not connected to ∞. For a reversible Markov chain this is the energy dissipated by the resistances. It is also the area of the resulting tiling. When the resistances are all rational, the Kirchhoff equations (and their generalizations (5)) are $\mathbb{Q}$-linear, and it makes sense to consider the energy not just as a number but as a *quadratic form* on $\mathbb{R}$ considered as a $\mathbb{Q}$-vector space.

9.1 SQUARES. Let P be a polygon which is tiled by rectangles of *arbitrary* aspect ratio. Associate to P the form $\theta(P) \in \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$:

$$\theta(P) = \sum_{rectangles} a_i \otimes b_i,$$

where the ith rectangle has horizontal side length a_i and vertical side length b_i.

LEMMA 11: *The form $\theta(P)$ is independent of the tiling.*

Proof: Any two rectangle tilings have a common subdivision (a rectangle tiling which refines each); yet such a refinement can be obtained by successively subdividing a single tile at a time with a horizontal or vertical edge: these subdivisions do not change θ. ∎

In particular if T can be tiled with squares then we can write θ in the form $\theta = \sum a_i \otimes a_i$, that is, θ is symmetric and positive semidefinite.

THEOREM 12: *If a rectilinear polygon P can be tiled with squares then θ is symmetric and positive semidefinite.*

This form θ is related to the response function that we discussed in section 8.1, but in general is a weaker invariant.

9.2 SQUARE-TILABLE TORI. In this section we compute the Euclidean tori which can be square-tiled.

Let T be a Euclidean torus, the quotient of $\mathbb{R}^2$ by a discrete lattice generated by two independent vectors (x, y) and (x', y'). Suppose that T is tiled with rectangles with sides parallel to the axes. As in the case of a polygon we can associate to this tiling a form θ: $\theta = \sum_{rectangles} a_i \otimes b_i$, which is independent of the rectangle tiling chosen by the same argument as in Lemma 11. A simple computation (see Figure 11) gives:

$$\theta = x \otimes y' - x' \otimes y.$$

THEOREM 13: *The torus T can be tiled with squares parallel to the axes if and only if θ is symmetric and positive semidefinite.*

Proof: The condition is necessary since θ can be written $\theta = \sum a_i \otimes a_i$. We prove sufficiency.

If θ is positive semidefinite then the four values x, y, x', y' must generate a $\mathbb{Q}$-vector subspace of $\mathbb{R}$ of dimension at most 2.

Firstly, if all four coordinate x, y, x', y' are in the same 1-dimensional $\mathbb{Q}$-vector subspace of $\mathbb{R}$ then it is easy to tile (an real homothety moves the lattice into $\mathbb{Z}^2$).

Secondly, if $y'/x \in \mathbb{Q} \cup \{\infty\}$ then we must have $y/x' \in \mathbb{Q} \cup \infty$ also, and if we are not in the previous case then positive semidefiniteness implies $x'y \leq 0$ and $xy' \geq 0$. Figure 11 shows a fundamental domain in the case $x, y, y' \geq 0 \geq x'$. The other choices of signs are similar.

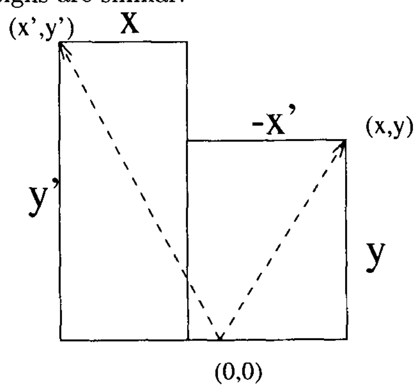

Figure 11.

Lastly, if $y'/x \notin \mathbb{Q} \cup \{\infty\}$ then we can write

$$x = r_1 y + r_2 y',$$
$$x' = r_3 y + r_4 y',$$

where $r_1, r_2, r_3, r_4 \in \mathbb{Q}$. The symmetry of θ forces $r_1 = -r_4$, and the positive semidefiniteness forces

$$r_2 \geq 0 \geq r_3 \tag{13}$$

and

$$-r_2 r_3 \geq r_1^2. \tag{14}$$

If $r_1 > 0$ then replace (x', y') with $(-x, -y)$ and (x, y) with (x', y'). This has the effect of replacing (r_1, r_2, r_3, r_4) with $(r_4, -r_3, -r_2, r_1)$. The new torus is tilable if and only if the old one is tilable, but now $r_1 < 0$.

If one of r_2 or $-r_3$ is less than $|r_1|$, consider the torus with lattice $(x, y)/m$ and $(x', y')/n$, where m, n are positive integers satisfying $\frac{m}{n} r_2 \geq |r_1|$ and $-\frac{n}{m} r_3 \geq |r_1|$.

Such m, n exist by (14). The original torus is an mn-fold cover of this new torus and the new torus satisfies $r_2, -r_3 \geq |r_1|$. If the new torus is square-tilable so is the old torus. Hence we may assume $r_2, -r_3 \geq |r_1|$.

Now Figure 12 shows a tiling of the torus with three rational rectangles, of aspect ratios $-r_1, -r_3 + r_1, r_2 + r_1$ which are all non-negative. ∎

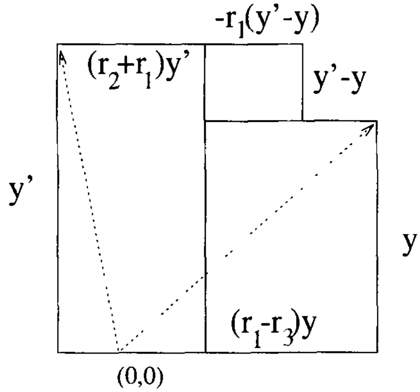

Figure 12.

Which tori are tilable in *some* direction? Let T_z, for z in the upper half plane $\{\Im(z) > 0\}$, be the quotient of $\mathbb{C}$ by the lattice generated by 1 and z. Up to scaling and rotation, any Euclidean torus is of the form T_z for some z. The complex number z is called the **modulus** of the torus.

THEOREM 14: *The torus T_z is square-tilable in some direction if and only if z lies on either: a circle, not crossing the x-axis, of rational center and rational radius, or: on a horizontal line at rational distance from the x-axis.*

Proof: Using the notation of the previous proof, assume we are in the generic situation $y'/x \notin \mathbb{Q} \cup \{\infty\}$, and $x = r_1 y + r_2 y'$, $x' = r_3 y - r_1 y'$ with

$$\begin{array}{ccccc} r_2 & \geq & 0 & \geq & r_3 \\ & -r_2 r_3 & \geq & r_1^2. & \end{array} \tag{15}$$

The associated torus has modulus

$$z = \frac{x' + iy'}{x + iy} = \frac{r_3 y - r_1 y' + iy'}{r_1 y + r_2 y' + iy}.$$

Let $t = y'/y \in \mathbb{R}$; this becomes

$$z(t) = \frac{r_3 + it - r_1 t}{r_1 + i + r_2 t} = \frac{i - r_1}{r_2} + \frac{r_2 r_3 + 1 + r_1^2}{r_2(r_1 + i + r_2 t)}.$$

As t runs over $\mathbb{R} \cup \{\infty\}$, $z(t)$ describes a circle. This circle has the property that the point with largest imaginary part is $\frac{i-r_1}{r_2}$ and the point with lowest imaginary part is

$$\frac{i-r_1}{r_2} - i\frac{r_2r_3+1+r_1^2}{r_2} = -\frac{r_1}{r_2} + i\left(\frac{-r_2r_3-r_1^2}{r_2}\right).$$

Hence the lowest point is not below the x-axis. Furthermore, any circle as in the statement uniquely determines r_1, r_2, r_3 satisfying (15). The remaining degenerate cases are similar and left to the reader. ∎

9.3 EQUILATERAL TRIANGLES. A similar "energy" form exists for equilateral triangle tilings. Let P be a polygon whose angles are multiples of $\pi/3$. Tile T by equilateral triangles and/or parallelograms having a $\pi/3$ angle. Associate to T an element $\theta \in \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$:

$$\theta = \sum_{triangles} a_i \otimes a_i + \sum_{parallelograms} b_j \otimes c_j + c_j \otimes b_j,$$

where a_i is the side length of the ith equilateral triangle and b_j, c_j are the side lengths of the j-th parallelogram.

LEMMA 15: *θ does not depend on the tiling.*

Proof: See the proof of Lemma 11. ∎

In particular if T can be tiled with equilateral triangles then $\theta = \sum a_i \otimes a_i$ is positive semidefinite.

THEOREM 16: *A polygon P can be tiled with equilateral triangles only if θ is positive semidefinite.*

Like the case of squares, this condition is not sufficient unless P is convex.

This theorem also gives a condition for a torus to be tilable; unlike the case of squares we do not know if the condition is also sufficient for the existence of a tiling of a torus.

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